# River networks on the slope-correlated landscape

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We study the morphologies of river networks on various landscapes. In general, the probability density distribution of drainage area $a$ of the river network scales as $P(a)\sim a^{-\tau}$. We consider a slope-slope correlation function $G(r)$ and define the persistent length $R$ where $G(r=R)$ becomes zero. In our restricted solid on solid network model, $R$ is independent of the system size $L$ and $\tau$ is close to 4/3, which is the value of the Scheidegger's river network model with random walk process. We also consider an avalanche model, where $R$ is proportional to $L$. There is a large slope-slope correlation length and the river network does not follow the directed random walk process with $\tau\approx 1.42$.

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River networks have been of a great deal of interest to both physicists and geologists [1,2]. The geological processes of the river networks display rich fractal structures. A river basin has an anisotropic structure that there are two length scales. One is a longitudinal length $L_\parallel$ which can be regarded as the system size $L$ and the other is a perpendicular length $L_\perp$. The perpendicular length scales as $L_\perp = L^H$ where $H$ is called the Hurst exponent [2]. If $H\neq 1$, the basin is a self-affine fractal and if $H=1$, it is a self similar fractal. The interesting quantity characterizing the properties of river basins is the area $a$ of drainage basin which is the number of sites connected to each other through drainage directions. The probability density distribution of drainage area $a$ is characterized by a scaling law [2]

$$p(a,L)=a^{-\tau}f(a/a_c(L)), \tag{1}$$

where $a_c(L)$ is a characteristic area defined by $a_c(L)\sim L^\phi$ with $\phi=1+H$. The scaling function $f(x)$ is a constant for $x\rightarrow 0$ and zero for $x\rightarrow\infty$. For $a\ll a_c(L)$, the distribution of drainage basin area thus scales as $p(a)\sim a^{-\tau}$. A lot of river network models [3,4] have a directed character due to the influence of a preferential flow direction. In this case, two exponents $\tau$ and $\phi$ are related by $\phi=(2-\tau)^{-1}$ [2]. Many observations in nature show the power law of $p(a)$ with $\tau=1.41\sim 1.45$ [2,5]. However $\tau$ is 4/3 in the Scheidegger model of the random walk system [3]. Here we consider two different models to understand on what conditions $\tau$ deviates from 4/3.

To describe the characteristics of landscapes, we consider both the surface width and the slope-slope correlation length. The surface width $W$, which is the root-mean square fluctuation of the surface height, is defined as

$$W(L,t)\equiv\left\langle \frac{1}{L^2}\sum_{\mathbf{x}}\left[h(\mathbf{x},t)-\overline{h}(t)\right]^2\right\rangle^{1/2}, \tag{2}$$

where $\overline{h}(t)$ denote the mean height at time $t$. The surface width follows a scaling behavior $W\sim L^\alpha g(t/L^z)$, where the scaling function $g(x)$ approaches to a constant for $x\gg 1$, and $g(x)\sim x^\beta$ for $x\ll 1$ with $z=\alpha/\beta$ [6]. The exponents $\alpha$, $\beta$ and $z$ are called the roughness, the growth, and the dynamic exponents, respectively.

The slope-slope correlation function is defined as $G(r,t)=\langle\nabla h(\mathbf{x}+r,t)\nabla h(\mathbf{x},t)\rangle$ where angular bracket indicates the average over $\mathbf{x}$. The slope-slope correlation length $R$ is obtained as the first zero of the slope-slope correlation function that $G[R(t),t]=0$. $R$ is the persistent length that the slope is correlated. It also can be regarded as the average mountain size of the system.

The statistical properties of river basins have a close relation with the morphology of the landscape. So, we study the connection between the probability distribution of basin area and the slope-slope correlation length of the landscape. The simplest river network can be developed by the random walk process. For example, in the Scheidegger's model [3], the drainage paths are in the direction of high gradients between watershed and main valley. The drainage direction is always downward but it may go to the left or to the right with equal probabilities. In the model, $H=1/2$ and $\tau=4/3$ is obtained due to the random walk process [3]. They are different from the measured values $H=0.75\sim 0.80$ and $\tau=1.41\sim 1.45$ [2,5] in real basins.

A large number of studies have been carried out to obtain various statistical properties of river networks and to construct models for the evolution of the drainage networks [1,3,4,7–11]. Some of them belong to the random walk class with $\tau=4/3$ [3,4]. The others show $\tau\approx 1.43$, which is similar to the results in real basins [9–11].

In this paper, we classify the river networks into two classes: the random walk class and the nonrandom walk class. First, we consider the restricted-solid-on-solid (RSOS) model [12] in which the slope-slope correlation length is short and independent of the system size $L$. After the surface width reaches at the saturated regime, we add a water drop of precipitation at every site and then allow the drop to flow down to the steepest descent direction to form a pattern of the river network on the landscape. The morphology of river network obtained on the landscape shows the value $H=0.5$ which is the same as the value of Scheidegger model (the

 

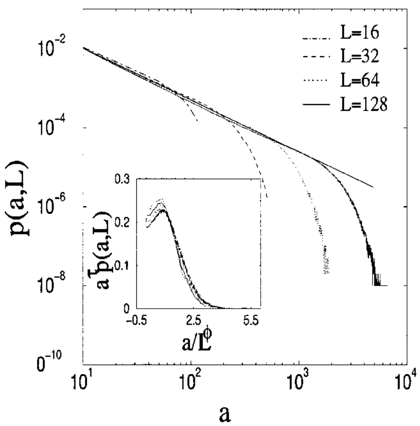


FIG. 1. The probability distribution $p(a,L)$ versus drainage area $a$ in log-log plot for the system sizes $L^2=16^2$, $32^2$, $64^2$, and $128^2$ in the RSOS river basin model. The straight guide line represents $\tau=1.32\pm0.01$. The inset shows the data collapse of $p(a,L)$ with $\tau=1.32$ and $\phi=1.47$.

random walk class). We also present a discrete model in which the slope is correlated up to the system size. In this case, the morphology of river network cannot be mapped into the random walk process.

We consider the RSOS model [12] which belongs to the Kadar-Paris-Zhang (KPZ) [13] universality class. The growth algorithm of the RSOS model is following: Select a site on two-dimensional square lattice randomly and deposit a particle provided the RSOS condition $|\Delta h|\leqslant 1$ is obeyed, where $\Delta h$ is the height difference on neighboring heights. For the RSOS model, it is well known that the roughness exponent $\alpha\approx0.4$ in $d=2+1$ and the slope-slope correlation length $R$ is independent of $L$ [12].

Once the surface width reaches the saturated regime, we add a water drop as a precipitation on the site $(x,y)$ in order to develop river networks. The drop is then allowed to flow to one of the three sites $(x-1,y-1)$, $(x,y-1)$, and $(x+1,y-1)$, causing a directed downward flow. A site with the smallest height is selected among them. Each drop thus flows down according to the steepest descent path until it reaches outlets $(y=0)$. The drainage area $a$ on a given site is then obtained by the area connected to the site which is the number of site connected to each other through the drainage direction. Periodic boundary condition is applied in the $x$ direction and free boundary condition in the $y$ direction.

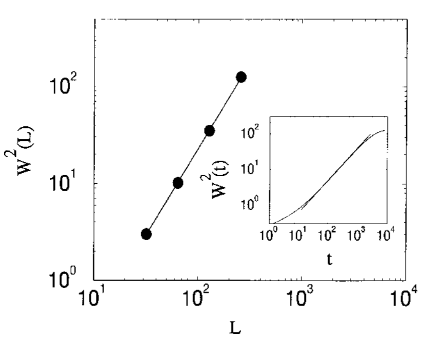


FIG. 2. The surface width $W^2(L)$ against system size $L$ is shown for $P=0.5$ for the system size $L^2=32^2$, $64^2$, $128^2$, and $256^2$. The guide line represents $2\alpha=1.78\pm0.01$. In the inset, $W^2(t)$ versus time $t$ are shown and the straight line is for $2\beta=0.89\pm0.01$.

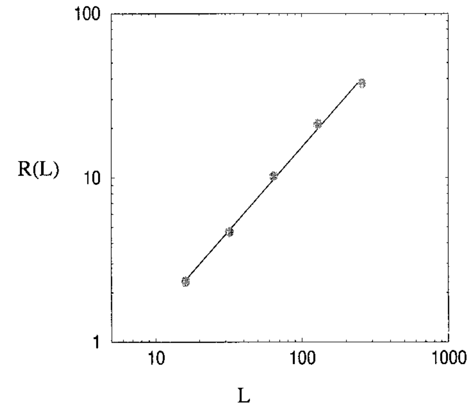


FIG. 3. The mound size $R(L)$ vs system size $L$ is shown for $L^2=16^2$, $32^2$, $64^2$, $128^2$, and $256^2$. The slope of the fitted line is $0.98\pm0.05$. This represents that the slope is correlated up to the system size.

The distribution of drainage basin area for different system sizes $L=16$, 32, 64, and 128 are shown in Fig. 1. The average is performed over 10 000 samples of different realizations in landscape and the statistics is taken over all river not over only the river with the largest flow at the outlet. The power law behavior with the exponent $\tau=1.32\pm0.01$ is observed. The data collapse with $\tau=1.32$ and $\phi=1.47$ is shown in the inset of Fig. 1. Since the water flow has a preferred direction due to the incline, the relation $\phi=(2-\tau)^{-1}$ for directed networks is well satisfied. These values are in good agreements with the values $\tau=4/3$ and $\phi=3/2$ obtained in Scheidegger's model [3]. Thus, if the slope-slope correlation length is finite, i.e., independent of the system size, we can treat the surface as the coarse grained height by the size of $R$. Then the river network can be mapped into the random walk process with $\tau=4/3$.

We also consider other discrete model which may mimic the erosion of soil material by water flow, avalanche of the soil, and the heterogeneity of the terrain. They can be taken as the essential ingrements of the model [14–16]. We preassign random numbers from zero to one to all sites on the flat substrate. They can represent the heterogeneity of soil materials, i.e., the erodibilities of materials. We then select a site randomly. If the random number on that site exceeds a pre-

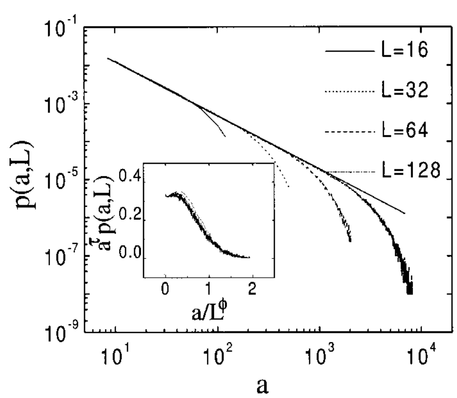


FIG. 4. The probability distribution of $p(a,L)$ versus drainage area $a$ for the system sizes $L^2=16^2$, $32^2$, $64^2$, and $128^2$ in the avalanche model. The straight guide line represents $\tau=1.42\pm0.01$. The inset shows the data collapse with $\tau=1.42$ and $\phi=1.72$.

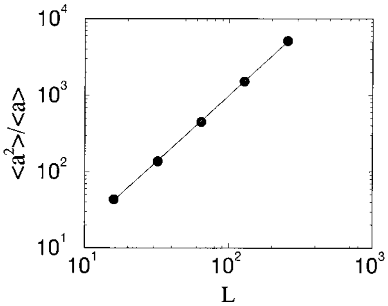


FIG. 5. Plot of the characteristic size of contributing area $\langle a^2\rangle/\langle a\rangle$ versus system size $L$ for $L^2=16^2$, $32^2$, $64^2$, $128^2$, and $256^2$. The slope of the fitted line is $1.73\pm0.01$.

assigned number $P$, a particle is deposited and slides down to the lowest site among the nearest neighbors. If the lowest site has the neighbor whose height is lower than that, it slides again until it reaches the site whose height is equal to or lower than the neighbor's heights. If the random number is less than $P$, a new site is selected randomly. We call it the avalanche model.

Our simulations are carried out starting from a flat initial condition with a periodic boundary condition in three dimensions. In Fig. 2, we have plotted the surface width $W^2(L)$ as a function of system size $L$ and $W^2(t)$ as a function of time $t$ for the avalanche model. The straight guide lines give us $\alpha=0.9\pm0.01$ and $\beta=0.45\pm0.01$. The used value of probability $P$ is 0.5, but qualitatively similar values of the exponents are obtained for any nonzero $P$. For simplicity, we restrict our simulations for the case $P=0.5$. We note that $\alpha\approx0.9$ of the model is close to that of the empirical results in nature [14,15,17,18].

Figure 3 shows the plot of the slope-slope correlation length $R(L)$ versus system size $L$. The slope of the fitted line is $0.98\pm0.05$. That is, in this avalanche model, the slope is correlated up to the system size. However, in other growth model such as Family model [19] and the RSOS model [12], $R$ is independent of $L$.

Once the surface width reaches the saturated regime, we develop a river network on the surface by the same water precipitation method of the RSOS model. Figure 4 shows a plot of $p(a,L)$ against drainage area $a$. The straight guide line represents $\tau=1.42\pm0.01$. The data collapse with $\tau=1.42$ and $\phi=1.72$ is shown in the inset of Fig. 4. To measure $\phi$ independently, we monitor the characteristic size of contributing area, $A$, defined by

$$A=\frac{\langle a^2\rangle}{\langle a\rangle}\sim L^{\phi}. \tag{3}$$

The relation can be provided from Eq. (1). Figure 5 shows a plot $\langle a^2\rangle/\langle a\rangle$ versus system size $L$. The fitted line represents $\phi=1.73\pm0.01$. With the value of $\tau$, it satisfys the relation $\phi=(2-\tau)^{-1}$ very well. These values are in an excellent agreement with the measurements in real river basins [5] and other models [1,9]. That is, the basin distribution function of the avalanche model is different from that of the random walk drainage basin model.

In Fig. 6, we show the morphologies of two landscapes and the corresponding river networks. The landscape obtained by the RSOS rule exhibits the configuration with many moderate hills and narrow valleys [see Fig. 6(a)]. It shows that the slope-slope correlation is very short. While the landscape obtained by the avalanche rule exhibits the configuration with few high mountains and broad but deep valleys [see Fig. 6(c)]. Typical patterns of river networks with $a\geqslant10$ are shown in Figs. 6(b) and 6(d). The morphology of the avalanche model is different from that of the RSOS model. In the avalanche model, the number of main

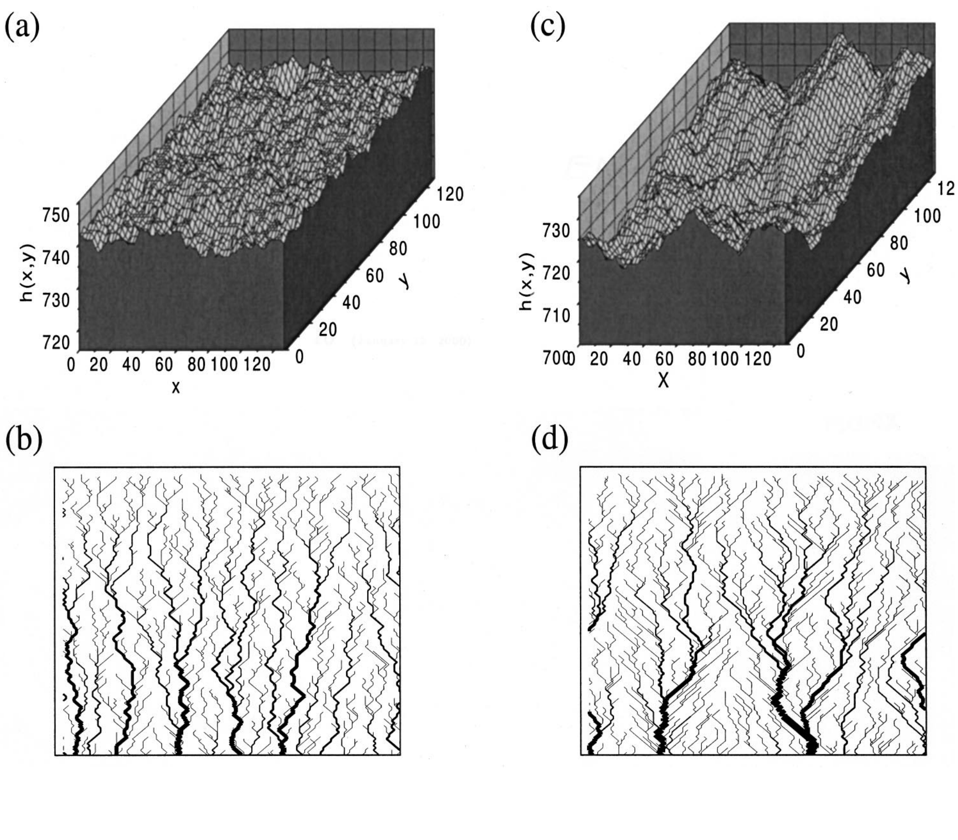


FIG. 6. (a) The landscape evolved by the RSOS model and (b) the corresponding river networks. (c) The landscape obtained by the avalanche model and (d) the corresponding river networks. The streams with $a\geqslant10$ are displayed for the system size $L^2=128\times128$.

streams covering the whole range of $y$ axes is small and the number of branches contributed to the main stream is large. They are thus distributed in relatively broaden range due to the existence of deep and broad valleys. These features are manifested by the large value of exponent $H$.

Empirically, it is expected that water flows to the downward direction along the side of mountain. Therefore, it may not follow the random walk process when water flows along the one tilted side of big mountains. If the slope-slope correlation length of a landscape is short, the system can be coarse grained by $R$ and the random walk process plays an important role in the formation of river network. However, if a system has long slope-slope correlation length, the development of river network cannot be achieved by the random walk process. In the system where the slope is correlated up to the system size, we expect that the pattern of river network deviates from the random walk process.

In summary, we have studied the river networks on two different models. The RSOS model has short slope-slope correlation length and the distribution of the drainage area $a$ scales as $p(a) \sim a^{-\tau}$ with $\tau \approx 1.32$. If we consider a coarse grained height by the slope-slope correlation length scale $R$, the network can be treated as the random walk process. So we insist that the network with short $R$ will have $\tau = 4/3$. We have also presented an avalanche model with long slope-slope correlation length which contains the features of random precipitation, avalanching, and erodibility of soils. In the system, the slope is correlated up to the system size and $\tau \approx 1.42$, which deviates from the value of the random walk process. In real river basins, $\tau = 1.41 \sim 1.45$ indicate that the formation of the river network does not follow the random walk process. This is due to the long slope-slope correlation length. It is interesting to measure the slope-slope correlation length in natural river basins.

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